Picard’s Little Theorem

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Regarding notation, \( \mathbb{H}^+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) denotes the upper half-plane. For \( \tau \in \mathbb{H}^+ \) and \( \Lambda = \mathbb{Z} + \tau : \mathbb{Z} \), the function \( \wp : \mathbb{C} \setminus \Lambda \rightarrow \mathbb{C} \) denotes the Weierstrass \( \wp \)-function given by

\[
\wp(z; \tau) = \frac{1}{z^2} + \sum_{(m, n) \neq (0, 0)} \frac{1}{(z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2}
\]

The modular function \( \lambda \) is defined by

\[
\lambda(\tau) = \frac{\wp \left( \frac{z + 1}{2}; \tau \right) - \wp \left( \frac{z}{2}; \tau \right)}{\wp \left( \frac{1}{2} \right) - \wp \left( \frac{1}{2} \right)} \quad \forall \tau \in \mathbb{H}^+
\]

Define (this is Ahlfors’ notation)

\[
\Omega = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2} \right| > \frac{1}{2}, 0 < \text{Re}(z) < 1 \right\}; \Omega' = \left\{ z \in \mathbb{C} : \left| z + \frac{1}{2} \right| > \frac{1}{2}, -1 < \text{Re}(z) < 0 \right\}
\]

Taking the closure in \( \mathbb{H}^+ \), we have the following result:

**Lemma 1.** \( \overline{\Omega} \cup \Omega' \) is a fundamental region for \( \Gamma(2) \).

**Proof.** See Theorem 8, pg. 281, Ahlfors’ Complex Analysis (Third Edition).

**Lemma 2.** The modular function \( \lambda \) maps \( \overline{\Omega} \cup \Omega' \) conformally onto \( \mathbb{C} \setminus \{0, 1\} \).


**Lemma 3.** The modular function \( \lambda : \mathbb{H}^+ \rightarrow \mathbb{C} \setminus \{0, 1\} \) is the universal cover of \( \mathbb{C} \setminus \{0, 1\} \).

**Proof.** We first remark that if \( \lambda : \mathbb{H}^+ \rightarrow \mathbb{C} \setminus \{0, 1\} \) is a covering space, then since \( \mathbb{H}^+ \) is convex and hence simply connected, \( \lambda : \mathbb{H}^+ \rightarrow \mathbb{C} \setminus \{0, 1\} \) is the universal cover of \( \mathbb{C} \setminus \{0, 1\} \).

Fix \( p \in \mathbb{C} \setminus \{0, 1\} \). Since \( \lambda \) maps the region \( \overline{\Omega} \cup \Omega' \) (where we use Ahlfors’ notation) conformally onto \( \mathbb{C} \setminus \{0, 1\} \), there exists a unique point \( w \in \overline{\Omega} \cup \Omega' \) such that \( \lambda(w) = p \). By adjusting our fundamental region, we may assume without loss of generality that \( w \) lies in interior of \( \overline{\Omega} \cup \Omega' \). Let \( U \ni w \) be an open neighborhood completely contained in the interior of \( \overline{\Omega} \cup \Omega' \). By the open mapping theorem, \( \lambda(U) = V \) is an open neighborhood of \( p \). Consider \( \lambda^{-1}(V) \subset \mathbb{H}^+ \). I claim that

\[
\lambda^{-1}(V) = \prod_{\gamma \in \Gamma(2)} \gamma \cdot U
\]

First observe that since \( U \) is contained in the interior of \( \overline{\Omega} \cup \Omega' \), \( \gamma \cdot U \cap U = \emptyset \) for \( \gamma \neq \text{Id} \in \Gamma(2) \). Recall that each point in \( \mathbb{H}^+ \) is equivalent under \( \Gamma(2) \) to exactly one point in \( \overline{\Omega} \cup \Omega' \). Hence,

\[
z \in \lambda^{-1}(V) \iff \gamma(z) \in \lambda^{-1}(V) \cap (\overline{\Omega} \cup \Omega') \quad \text{for some} \ \gamma \in \Gamma(2) \iff \gamma(z) \in U
\]

since \( \lambda \) is a bijective on \( \overline{\Omega} \cup \Omega' \). Linear fractional transformations are automorphisms, hence \( \gamma \cdot U \) is open \( \forall \gamma \in \Gamma(2) \). \( \lambda : U \rightarrow V \) is a continuous, bijective open map and therefore a homeomorphism. Since \( \lambda \) is invariant under \( \Gamma(2) \), we conclude that \( \lambda \) maps \( \gamma \cdot U \) homeomorphically onto \( V \) \( \forall \gamma \in \Gamma(2) \).

We will need the following topological lemma:

**Lemma 4.** Suppose that \( p : X \rightarrow Y \) is a covering map, \( D \) is a path-connected, locally path-connected and simply connected topological space, and \( f : D \rightarrow Y \) is continuous. Suppose that \( a \in D \). Fix \( x_0 \in X \) with \( p(x_0) = f(a) \). Then there exists a unique continuous function \( \tilde{f} : D \rightarrow X \) such that \( f(a) = x_0 \) and \( p \circ \tilde{f} = f \).

**Proof.** See Lemma 79.1 (General Lifting Lemma), pg. 479 in Munkres’ Topology (Second Edition).
Lemma 5. Let $\Omega, V \subset \mathbb{C}$ be regions and $p : \Omega \to V$ be a holomorphic covering map. If $f : \mathbb{C} \to V$ is holomorphic, then for each $a \in \mathbb{C}$ and $z \in \Omega$ such that $p(z) = f(a)$, there exists a unique holomorphic function $\tilde{f} : \mathbb{C} \to \Omega$ such that $\tilde{f}(a) = z$ and $p \circ \tilde{f} = f$.

Proof. The preceding lemma tells us that there is a unique continuous function $\tilde{f} : \mathbb{C} \to V$ such that $\tilde{f}(a) = z$ and $p \circ \tilde{f} = f$. We show that $\tilde{f}$ is holomorphic. Fix $w \in \mathbb{C}$. Since $p$ is a covering map, there exists an open neighborhood $W \ni f(w)$ whose preimage is the disjoint union of open sets in $\Omega$, each mapped homeomorphically onto $W$. In particular, there exists an open neighborhood $U \ni \tilde{f}(w)$, such that $p : U \to W$ is conformal. Hence, $p$ has a local holomorphic inverse $p^{-1} : W \to U$. The open mapping theorem and the fact that $U$ is open tells us that $D(\tilde{f}(w); r) \subset U$ for $r > 0$ sufficiently small. Therefore

$$p^{-1} \circ f(w') = p^{-1} \circ (p \circ \tilde{f})(w') = \tilde{f}(w') \forall w' \in \tilde{f}^{-1}(D(\tilde{f}(w); r))$$

Since $\tilde{f}$ is continuous, $\tilde{f}^{-1}(D(\tilde{f}(w); r))$ is an open neighborhood of $w$. We conclude that $\tilde{f}$ is holomorphic in a neighborhood of $w$. Since $w \in \mathbb{C}$ was arbitrary, we conclude that $\tilde{f}$ is holomorphic.

Theorem 6. (Picard) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. If there exist $w_1 \neq w_2 \in \mathbb{C}$ such that $f(\mathbb{C}) \subset \mathbb{C} \setminus \{w_1, w_2\}$, then $f$ is constant.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function which omits two values $w_1 \neq w_2$. Replacing $f$ by

$$g(z) = \frac{f(z) - w_1}{w_2 - w_1}$$

we may assume that $f(\mathbb{C}) \subset \mathbb{C} \setminus \{0, 1\}$. Since $\lambda : \mathbb{H}^+ \to \mathbb{C} \setminus \{0, 1\}$ is the universal cover of $\mathbb{C} \setminus \{0, 1\}$, the preceding lemma tells us that $f$ lifts to a map $\tilde{f} : \mathbb{C} \to \mathbb{H}^+$ such that $\lambda \circ \tilde{f} = f$. Let $\psi : \mathbb{H}^+ \to \mathbb{D}$ be a conformal map from the upper-half plane onto the unit disk. Then $\psi \circ \tilde{f}$ is a bounded entire function and therefore constant by Liouville’s theorem. Since $\psi$ is bijective, $\tilde{f}$ is constant, from which we conclude that $f$ is constant.

$\square$