Monty Hall Problem

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Consider a nightly gameshow with the following setup. There are three doors. Behind each of two of the doors is a goat. Behind the third remaining door is a car. On any given night, the car is 'equally likely' to be behind any one of the three doors. There are a host, who knows the location of the car, and there is a contestant, who does not know the location of the car. The contestant chooses a door. The host then opens one of the doors, behind which is a goat, and offers the contestant the chance two swap his or her door for the remaining door.

Should the contestant keep his or her door or swap it for the other?

We offer a mathematical model of this problem. Label the doors $1, 2, 3$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $C : \Omega \to \{1, 2, 3\}$ be the random variable representing the location of the car.

$$\mathbb{P}(C = 1) = \mathbb{P}(C = 2) = \mathbb{P}(C = 3) = \frac{1}{3}$$

Let $S : \Omega \to \{1, 2, 3\}$ be the random variable representing the contestant's door selection. We assume that C and S are independent Let $H : \Omega \to \{1, 2, 3\}$ be the random variable corresponding to the host's choice of door to open. We make the following assumptions on H. The host is 'equally likely' to choose between the elements of $\{1, 2, 3\} \setminus \{C, S\}$.

Suppose $C = 1$. We use Bayes' rule to compute the probability of winning by switching.

$$\begin{aligned}
\mathbb{P}(C = 2 | S = 1, H = 3) = \frac{\mathbb{P}(C = 2, S = 1, H = 3)}{\mathbb{P}(S = 1, H = 3)} &= \frac{\mathbb{P}(H = 3 | C = 2, S = 1)\mathbb{P}(C = 2, S = 1)}{\sum_{j=1}^{3} \mathbb{P}(S = 1, H = 3 | C = j)\mathbb{P}(C = j)} \\
&= \frac{\mathbb{P}(H = 3 | C = 2, S = 1)\mathbb{P}(C = 2 | S = 1)\mathbb{P}(S = 1)}{\sum_{j=1}^{3} \mathbb{P}(H = 3 | S = 1, C = j)\mathbb{P}(S = 1)\mathbb{P}(C = j)} \\
&= \frac{1 \cdot \frac{1}{3} \cdot \mathbb{P}(S = 1)}{\mathbb{P}(S = 1) \sum_{j=1}^{3} \mathbb{P}(H = 3 | S = 1, C = j)\mathbb{P}(C = j)} \\
&= \frac{\frac{1}{3}}{\left(\frac{1}{2} \cdot \frac{1}{3}\right) + \left(1 \cdot \frac{1}{3}\right) + \left(0 \cdot \frac{1}{3}\right)} \\
&= \frac{2}{3}
\end{aligned}$$

By symmetry, we see that

$$\begin{aligned}
\mathbb{P}(C = 3 | S = 1, H = 2) = \mathbb{P}(C = 1 | S = 2, H = 3) = \mathbb{P}(C = 3 | S = 2, H = 2) = \mathbb{P}(C = 1 | S = 3, H = 2) \\
= \mathbb{P}(C = 2 | S = 3, H = 1) = \frac{2}{3}
\end{aligned}$$

We conclude that the contestant is always better off swapping doors.