Definition 1. A generalized (complex) linear normed space \((X, \| \cdot \|)\) is a (complex) vector space \(X\) with a function \(\| \cdot \| : X \to \mathbb{R}_{\geq 0}\) satisfying

1. \(\|x\| = 0 \iff x = 0;\)
2. \(\|x + y\| \leq \|x\| + \|y\|;\)
3. \(\|\alpha x\| \to 0\) if \(\alpha \to 0\) and \(\|ix\| = \|x\|\).

Proposition 2. (Jordan-von Neumann theorem) Let \((X, \| \cdot \|)\) be a generalized linear normed space. Then there exists an inner product \((\cdot, \cdot) : X \times X \to \mathbb{C}\) such that \(\|x\| = \sqrt{(x, x)}\) if and only if

\[
\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y\|^2\right) \quad \forall x, y \in X
\]

In this case, the inner product \((\cdot, \cdot)\) is uniquely determined by the preceding requirement.

Proof. We first prove the \(\Rightarrow\) direction. Then, for \(x, y \in X,\)

\[
\|x + y\|^2 + \|x - y\|^2 = (x + y, x + y) + (x - y, x - y) = (x, x) + (x, y) + (y, y) + (x, y) + (x, x) - (y, x) - (y, x) + (y, y)
\]

\[
= 2 \left(\|x\|^2 + \|y\|^2\right),
\]

which shows that the condition of the theorem is necessary. Also,

\[
\|x + y\|^2 - \|x - y\|^2 = (x + y, x + y) - (x - y, x - y) = (x, x) + (x, y) + (y, y) + (x, y) + (x, x) - (y, x) - (y, y)
\]

\[
= 2 [(x, y) + (y, x)]
\]

\[
= 4 \Re(x, y)
\]

Replacing \(x\) by \(ix\) and using the property that \((ix, y) = i(x, y)\) and therefore \(\Re i(x, y) = -\Im(x, y)\), we obtain

\[
\|ix + y\|^2 - \|ix - y\|^2 = -4 \Im(x, y)
\]

Thus,

\[
\Re(x, y) = \frac{1}{4} \left(\|x + y\|^2 + \|x - y\|^2\right), \quad \Im(x, y) = \frac{1}{4} \left(\|ix - y\|^2 - \|ix + y\|^2\right),
\]

which shows that \((\cdot, \cdot)\) is uniquely determined by \(\| \cdot \|\).

We now prove the \(\Leftarrow\) direction. Define \((\cdot, \cdot) : X \times X \to \mathbb{C}\) by

\[
\begin{cases}
\Re(x, y) = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2\right) \\
(x, y) = \Re(x, y) - i \Re(ix, y)
\end{cases}
\]

We now show that \((\cdot, \cdot)\) satisfies the axioms of an inner product. Let \(x, x', y, y' \in X\). Then

\[
2 \left(\|x + y\|^2 - \|x - y\|^2\right) = 2 \left(\|x + y\|^2 + \|x'\|^2\right) - 2 \left(\|x - y\|^2 + \|x'\|^2\right)
\]

\[
= \|x + y\|^2 + \|x + y\|^2 - \|x - y\|^2 + \|x - y\|^2 - \left(\|x - y\|^2 + \|x - y\|^2\right)
\]

\[
= \left(\|x + x'\| + \|y\|^2 - \|x + x'\| - \|y\|^2\right) + \left(\|x - x'\| + \|y\|^2 - \|x - x'\| - \|y\|^2\right)
\]
Hence,

\[ 8 \text{Re}(x, y) = 4 \left[ \text{Re}(x + x', y) + \text{Re}(x - x', y) \right] \iff 2 \text{Re}(x, y) = \text{Re}(x + x', y) + \text{Re}(x - x', y) \]

Using our hypotheses that \( \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \) with \( x = 0, y \in X \), we obtain that \( \|y\| = \|-y\| \)
for all \( y \in X \). Hence, \( \text{Re}(0, y) = 0 \). Taking \( x = x' \), we obtain that

\[ 2 \text{Re}(x, y) = \text{Re}(2x, y) + \text{Re}(0, y) = \text{Re}(2x, y) \]

Replacing \( x \) and \( x' \) by \( \frac{1}{2}(x + x') \) and \( \frac{1}{2}(x - x') \), respectively, we see that

\[ \text{Re}(x + x', y) = 2 \text{Re}(\frac{1}{2}(x + x'), y) = \text{Re}(\frac{1}{2}(x + x') + \frac{1}{2}(x - x'), y) + \text{Re}(\frac{1}{2}(x + x') - \frac{1}{2}(x - x'), y) \]

By definition of \( (\cdot, \cdot) \),

\[ (x + x', y) = \text{Re}(x + x', y) - i \text{Re}(ix + ix', y) = \text{Re}(x, y) + \text{Re}(x', y) - i \text{Re}(ix, y) - i \text{Re}(ix', y) = (x, y) + (x', y) \]

Using the triangle inequality property of \( \|\cdot\| \) and the result that \( \|x\| = \|-x\| \) \( \forall x \in X \), we obtain that

\[ \|\alpha x + y\| - \|\beta x + y\| \leq \|\alpha - \beta\| x \| \]

By hypothesis that \( \|\cdot\| \) is a generalized norm, we have that \( \alpha \rightarrow \beta \Rightarrow \|\alpha x + y\| \rightarrow \|\beta x + y\| \). From the definition of \( \text{Re}(\alpha x, y) \), we see that \( \text{Re}(\alpha x, y) \) is continuous in \( x \), which implies that \( (\alpha x, y) \) is continuous in \( x \).

Let \( S := \{ \alpha \in \mathbb{C} : (\alpha x, y) = \alpha(x, y) \forall x, y \in X \} \). Clearly, 0, 1 \( \in \) \( S \). Furthermore, it is clear that \( \alpha, \beta \in \mathbb{S} \Rightarrow \alpha \pm \beta \in \mathbb{C} \), which implies that \( \mathbb{Z} \subseteq \mathbb{S} \). I now claim that \( \mathbb{Q} \subseteq \mathbb{S} \). Let \( \alpha, \beta \in \mathbb{Z} \) with \( \beta \neq 0 \). Since \( \alpha, \beta \in \mathbb{S} \), we see that

\[ (\alpha x, y) = (\alpha x, y) = (\beta \cdot \frac{\alpha}{\beta} x, y) = \beta(\frac{\alpha}{\beta} x, y) = (\frac{\alpha}{\beta} x, y) \quad \forall x, y \in X, \]

which completes the proof of the claim. Since \( (\alpha x, y) \) is continuous in \( x \), for fixed \( x, y \in X \), we conclude from the density of \( \mathbb{Q} \) in \( \mathbb{R} \) that \( \mathbb{Q} \subseteq \mathbb{S} \). To see that \( \mathbb{C} \subseteq \mathbb{S} \), which implies equality, observe that

\[ (ix, y) = \text{Re}(ix, y) - i \text{Re}(i \cdot ix, y) = \text{Re}(x, y) - i \text{Re}(x, y) = \text{Re}(x, y) + i \text{Re}(x, y) = i \left[ -i \text{Re}(x, y) + \text{Re}(x, y) \right] = i(x, y) \]

Noting \( (\alpha)x = \alpha(x) \), for \( \alpha \in \mathbb{R} \) and \( x \in X \), completes the proof.

Since \( \|ix\| = \|x\| \) for all \( x \in X \) by definition of a generalized norm, we have that

\[ \text{Re}(ix, iy) = \frac{1}{4} \left( \|ix + iy\|^2 - \|ix - iy\|^2 \right) = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) = \text{Re}(y, x) \]

and

\[ \text{Re}(x, y) = \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) = \frac{1}{4} \left( \|y + x\|^2 - \|y - x\|^2 \right) = \text{Re}(y, x) \]

Hence,

\[ \text{Re}(ix, y) = \text{Re}(i \cdot ix, iy) = \text{Re}(-x, iy) = - \text{Re}(x, iy) = - \frac{1}{4} \left( \|x + iy\|^2 - \|x - iy\|^2 \right) \]

\[ = - \frac{1}{4} \left( \|ix + y\|^2 - \|-ix - y\|^2 \right) \]

\[ = - \frac{1}{4} \left( \|y - ix\|^2 - \|y + ix\|^2 \right) \]

\[ = \text{Re}(y, ix) \]

We conclude that

\[ (y, x) = \text{Re}(y, x) + i \text{Re}(iy, x) = \text{Re}(x, y) + i \text{Re}(x, iy) = \text{Re}(x, y) + i \text{Re}(ix, y) = \text{Re}(x, y) - i \text{Re}(ix, y) = (x, y) \]
As a consequence, we obtain
\[(\alpha x, \alpha x) = \alpha(x, \alpha x) = \alpha(x, x) = |\alpha|^2 (x, x), \quad \forall \alpha \in \mathbb{C}, x \in X\]

We complete the proof of the theorem by showing that \(\|x\| = \sqrt{x, x}\). Observe that
\[(x, x) = \Re(x, x) - i \Re(i x, x) = \frac{1}{4} \left(\|x + x\|^2 - \|x - x\|^2\right) - i \frac{1}{4} \left(\|i x + x\|^2 - \|i x - x\|^2\right)
= \|x\|^2 - i \frac{1}{4} \left(\|i x + x\|^2 - |i|^2 \|x + i x\|^2\right)
= \|x\|^2
\]

\[\Box\]

**Proposition 3.** Let \((X, \|\cdot\|)\) be a linear normed space. Define
\[C_{x,y} := \frac{1}{2} \frac{\|x + y\|^2 + \|x - y\|^2}{\|x\|^2 + \|y\|^2}, \quad \forall x, y \in X, \text{ not } x = y = 0\]

Define \(b := \sup_{x,y \in X} C_{x,y}\) and \(a := \inf_{x,y \in X} C_{x,y}\). Then \(\frac{1}{2} \leq a \leq 1 \leq b \leq 2\) and moreover, \(a = \frac{1}{b}\). A linear normed space has an inner product if and only if \(a = b = 1\).

**Proof.** The last assertion is just Jordan-von Neumann theorem. For the string of inequalities, observe that by the triangle inequality
\[\frac{1}{2} \frac{\|x + y\|^2 + \|x - y\|^2}{\|x\|^2 + \|y\|^2} \leq \frac{\|x\|^2 + 2 \|x\| \|y\| + \|y\|^2}{\|x\|^2 + \|y\|^2} \leq \frac{2(\|x\|^2 + \|y\|^2)}{\|x\|^2 + \|y\|^2} = 2,
\]

and by the reverse triangle inequality,
\[\frac{1}{2} \frac{\|x + y\|^2 + \|x - y\|^2}{\|x\|^2 + \|y\|^2} = 2 \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x + y\| + \|x - y\|)^2} \geq \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x + y\|^2 + \|x - y\|^2)} \geq \frac{\|x + y\|^2 + \|x - y\|^2}{2((\|x + y\|^2 + \|x - y\|^2)} = \frac{1}{2}
\]

To see that \(a = \frac{1}{b}\), observe that
\[C_{x+y,x-y} = \frac{1}{2} \frac{\|2x\|^2 + \|2y\|^2}{\|x + y\|^2 + \|x - y\|^2} = 2 \frac{\|x\|^2 + \|y\|^2}{\|x + y\|^2 + \|x - y\|^2} = 1 \frac{C_{x,y}}{C_{x,y}}
\]

\[\Box\]

**References**