Hörmander's Staircase

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Abstract

We construct the function known as Hörmander's staircase used in his proof of the Malgrange-Ehrenpreis theorem, which states that every non-zero linear differential operator with constant coefficients has a fundamental solution.

We will need the following lemma from complex analysis.

Lemma 1. *Let $P(z_1, \cdots, z_d)$ be a polynomial in $\mathbb{C}^d$ such that for any fixed $w = (z_2, \cdots, z_d)$, the polynomial*

$$p_w(z) = P(z, z_2, \cdots, z_d)$$

has degree m. Then the roots $\alpha_1(w), \cdots, \alpha_m(w)$ of $p_w(z)$ are continuous functions of w.

Proof. Fix $w_0 \in \mathbb{C}^{d-1}$, and let $\epsilon > 0$ be given. Replacing ϵ a smaller $\epsilon' > 0$ if necessary, we may assume that $\overline{D}(\alpha_i(w_0)) \cap \overline{D}(\alpha_j(w_0)) = \emptyset$ for all $i \neq j$. Since the argument will be the same for each $\alpha_j(w)$, it suffices to show that $\alpha_1(w)$ is continuous at $w = w_0$. Since $p_{w_0}(z)$ vanishes nowhere on $\overline{D}(\alpha_1(w_0); \epsilon) \setminus \{\alpha_1(w_0)\}$, $|p_{w_0}(z)|$ attains its minimum at some $z_1 \in \partial D(\alpha_1(w_0); \epsilon)$ and $|p_{w_0}(z)| > 0$. Choose $\delta > 0$ such that

$$|(z_1, \cdots, z_d) - (z_1', \cdots, z_d')| < \delta \Rightarrow |P(z_1, \cdots, z_d) - P(z_1', \cdots, z_d)| < \frac{p_{w_0}(z_1)}{2}$$

We want to show that $p_w(z)$ and $p_{w_0}(z)$ have the same number of roots for all $|w - w_0| < \delta$. Consider the polynomial $q_w(z)$ defined by

$$q_w(z) = p_w(z) - p_{w_0}(z)$$

Since

$$|q_w(z)| = |p_w(z) - p_{w_0}(z)| = |P(z, w) - P(z, w_0)| < \frac{p_{w_0}(z)}{2} = p_{w_0}(z)$$

for all $|z - \alpha_1(w_0)| = \epsilon$, it follows from Rouché's theorem that $p_{w_0}(z)$ and $q_w(z) + p_{w_0}(z) = p_w(z)$ have the same number of roots in $D(\alpha_1(w_0); \epsilon)$. Hence, $p_w(z)$ has one root in $D(\alpha_1(w_0); \epsilon)$, or equivalently,

$$w, w_0 \in \mathbb{C}^{d-1} \, |w - w_0| < \delta \Rightarrow |\alpha_1(w) - \alpha_{w_0}(w)| < \epsilon$$

□

Proposition 2. *(Hörmander's Staircase) There exists a function $n : \mathbb{R}^{d-1} \to \mathbb{Z}, \xi' \mapsto n(\xi')$ such that*

1. *$|n(\xi')| \leq m + 1$ for all ξ'.*
2. *If $\mathrm{Im}(\xi_1) = n(\xi')$, then $|\xi_1 - \alpha_j(\xi')| \geq 1$ for $j = 1, \cdots, m$.*
3. *The function n is Borel measurable.*

Proof. Recall that $p(z) = P(z, \xi')$, which has m roots. Consider the disjoint intervals

$$I_l := (-m - 1 + 2l, -m - 1 + 2(l + 1)], \qquad l = 0, \cdots, m$$

Since there are $m + 1$ disjoint intervals and m points $\alpha_1(\xi'), \cdots, \alpha_m(\xi')$, there exists l such that

$$\mathrm{Im}(\alpha_i(\xi')) \notin I_l, \qquad \forall i = 1, \cdots, m$$

Choose $n(\xi')$ to be the midpoint of this I_l. This gives us property (2). Clearly, $|n(\xi')| \leq m + 1$ for all ξ', which gives us property (1) . For measurability, the preceding lemma tells us that the functions $\xi' \mapsto \alpha_i(\xi')$ are continuous from $\mathbb{C}^{d-1} \to \mathbb{C}$. □