Baire Category and Measure

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Definition 1. We say that a subset $S \subset \mathbb{R}$ has **measure zero** or is a **null set** if for any $\epsilon > 0$, there exists a countable collection of open (it doesn't really matter) intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$S \subset \bigcup_{j=1}^{\infty} I_j \text{ and } \sum_{j=1}^{\infty} |I_j| < \epsilon,$$

where, if $I_j = (a_j, b_j)$, $|I_j| := b_j - a_j$.

Proposition 2. *The countable collection of null sets is again a null set.*

Proof. Suppose we have a countable collection $\{A_n\}_{n=1}^{\infty}$ of null sets, and let $\epsilon > 0$ be given. By definition of measure zero, for each $n \in \mathbb{Z}^{\geq 1}$, there exists a countable collection of open intervals $\{I_{n,j}\}_{j=1}^{\infty}$ such that

$$A_n \subset \bigcup_{j=1}^{\infty} I_{n,j} \text{ and } \sum_{j=1}^{\infty} |I_{n,j}| < \frac{\epsilon}{2^n}$$

Since the countable union of a countable set is itself countable, we have that $\{I_{n,j} : 1 \leq n \leq \infty, 1 \leq j \leq \infty\}$ is a countable collection of open intervls. Summing a geometric series, we obtain that

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} I_{n,j} \text{ and } \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |I_{n,j}| < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

□

Exercise 1. *Show that any nonempty open subset of $\mathbb{R}$ is not a null set.*

Definition 3. A property P of real numbers is said to hold **almost everywhere** or to hold for **almost all** x if the S, where the property P does not hold, has measure zero.

Definition 4. A **Baire space** is a topological space (X, τ) with the property that the countably intersection of dense open sets is again dense. We say that a set A in a topological space (X, τ) is **nowhere dense** if $X \setminus \overline{A}$, where $\overline{A}$ denotes the closure of A in X, is a dense open set. Equivalently, A is nowhere dense if it has empty interior. If A is the countable union of nowhere dense sets, then we say that A is of **first category** or is **meage**. If A is not of first category, then we say that A is of **second category** or **nonmeagre**.

We say that a property P holds **quasi-everywhere** in a topological space if the set of points where P does not hold is meagre.

The reader can easily verify that the countable union of meagre sets is again meagre.

Theorem 5. *(Baire) Let (X, d) be a complete metric space. Then X endowed with the metric topology is a Baire space.*

Proof. Let $(O_n)_{n=1}^{\infty}$ be a countable collection of dense open sets. Let $B \subset X$ be an arbitrary open ball. We will show that $B \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset$. Since O_1 is dense, $B \cap O_1 \neq \emptyset$ and is open. Thus, there exists a ball B_1 such that

$$B_1 \subset \overline{B}_1 \subset B \cap O_1$$

Suppose we have chosen open balls $B_1, \cdots, B_n$ such that

$$\overline{B}_{j+1} \subset B_j \cap O_{j+1}$$

Since O_{n+1} is dense, $B_n \cap O_{n+1} \neq \emptyset$ is open. Hence, there exists an open ball B_{n+1} such that

$$B_{n+1} \subset \overline{B}_{n+1} \subset B_n \cap O_{n+1}$$

By induction, we obtain a shrinking sequence of open balls $(B_n)_{n=1}^{\infty}$ such that $\overline{B}_{n+1} \subset B_n$. I claim that $\bigcap_{n=1}^{\infty} \overline{B}_n \neq \emptyset$. Without loss of generality, we may assume that $\text{diam}(B_n) \to 0, n \to \infty$. For each n, let $(x_n)_{n \in \mathbb{N}}$ be a sequence such that $x_n \in B_n$. Then (x_n) is Cauchy and by completeness of X, there exists $x \in X$ such that $x_n \to x$. It is immediate that $x \in \bigcap_{n=1}^{\infty} \overline{B}_n$. By construction, $x \in B \cap \bigcap_{n=1}^{\infty} O_n$. □

We now show that if our topological space is $\mathbb{R}$ equipped with the Euclidean metric topology, then "quas-everywhere" is not equivalent to "almost everywhere."

Theorem 6. *There exists a property P about real numbers such P that holds quasi-everywhere and $\neg P$ holds almost everywhere.*

Proof. Since we can take the property P to be $x \in A$ for some set $A \subset \mathbb{R}$, it suffices to show that there exists a subset $A \subset \mathbb{R}$ such that A has measure zero and $\mathbb{R} \setminus A$ is meagre. Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q}$. For each $(i,j) \in \mathbb{Z}^{\geq 1} \times \mathbb{Z}^{\geq 1}$, let

$$I_{ij} := \left(a_i - \frac{1}{2^{i+j+1}}, a_i + \frac{1}{2^{i+j+1}}\right)$$

Define a countable collection of open sets $(O_j)_{j=1}^{\infty}$ by

$$O_j := \bigcup_{i=1}^{\infty} I_{ij}$$

I claim that $\bigcap_{j=1}^{\infty} O_j$ has measure zero. Indeed, for any $k \in \mathbb{Z}^{\geq 1}$,

$$\bigcap_{j=1}^{\infty} O_j \subset O_k = \bigcup_{j=1}^{\infty} I_{jk} \text{ and } \sum_{j=1}^{\infty} |I_{jk}| = \sum_{j=1}^{\infty} \frac{1}{2^{k+j}} = \frac{1}{2^k} \to 0, k \to \infty$$

Set $F_j := X \setminus O_j$, which is tautologically closed. I claim that F_j is nowhere dense for each $j \in \mathbb{Z}^{\geq 1}$. Otherwise by the density of $\mathbb{Q}$ in $\mathbb{R}$, $\emptyset \neq (x-r, x+r) \cap \mathbb{Q} \subset F_j \cap \mathbb{Q}$ for some $x \in F_j$, which is a contradiction. Thus, $\bigcup_{j=1}^{\infty} F_j$ is meagre. Setting $A := \bigcap_{j=1}^{\infty} O_j$ gives us the desired set. $\square$