Uncountability of Hamel Basis for Banach Space

Matt Rosenzweig

A remark on notation: for a normed space \((X, \|\cdot\|)\), we denote the open ball around an element \(x \in X\) of radius \(r > 0\) by \(B(x; r)\). We denote the closed ball by \(\overline{B}(x; r)\).

**Lemma 1.** Let \((X, \|\cdot\|)\) be a normed space and let \(V \subset X\) be a proper closed subspace. Then \(V\) is nowhere dense, or equivalently, \(X \setminus V\) is a dense open set.

**Proof.** Assume the contrary, so that \(V\) has nonempty interior. Hence, there exists \(v \in V\) and \(r > 0\) such that \(B(v; r) \subset V\). Then
\[
v + \frac{x - v}{2r \|x - v\|} \in B(v; r) \subset V, \quad \forall x \in X \setminus \{v\}
\]
Since \(V\) is a closed under linear combinations, for any \(x \neq v\),
\[
x = 2r \|x - v\| \left( v + \frac{x - v}{2r \|x - v\|} \right) + (2r \|x - v\| - 1) v \in V,
\]
which contradicts that \(V\) is a proper subspace of \(X\).

**Definition 2.** Let \(X\) be a vector space over some field \(F\). We say that a set \(B \subset X\) is a Hamel basis for \(X\) if \(\text{span}_F(B) = X\) and any finite subset \(\{x_1, \cdots, x_n\} \subset B\) is linearly independent.

**Proposition 3.** Let \((X, \|\cdot\|)\) be an infinite-dimensional Banach space, and \(B\) be a Hamel basis for \(X\). Then \(B\) is uncountable.

**Proof.** Suppose \(B\) is countable, and let \(\{x_n : n \in \mathbb{Z}^{\geq 1}\}\) be an enumeration of \(B\). For each \(n \in \mathbb{Z}^{\geq 1}\), consider the sets \(E_n := \text{span}_F \{x_1, \cdots, x_n\}\). It is well-known that any finite-dimensional subspace of a Banach space is closed, hence \(E_n\) is closed. Furthermore, since \(B\) is a Hamel basis, for all \(x \in X\), we can write \(x = \sum_{j=1}^n c_j x_j\), where we allow \(c_j = 0\), for some \(n\). Thus, \(x \in E_n\) for some \(n\), which implies that \(X = \bigcup_{n=1}^\infty E_n\).

Since \(X\) is infinite-dimensional, \(E_n \subset X\). Since \(E_n\) is closed, we have by the preceding lemma that \(E_n\) is nowhere dense in \(X\). Hence, \(X\) is the countable union of nowhere dense sets. Since \(X\) is complete, we obtain a contradiction by the Baire category theorem.

\[\square\]